

STURM-LIOUVILLE OPERATORS WITH MATRIX DISTRIBUTIONAL COEFFICIENTS

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ABSTRACT. The paper deals with the singular Sturm-Liouville expressions

$$l(y) = -(py')' + qy$$

with the matrix-valued coefficients p, q such that

$$q = Q', \quad p^{-1}, \quad p^{-1}Q, \quad Qp^{-1}, \quad Qp^{-1}Q \in L_1,$$

where the derivative of the function Q is understood in the sense of distributions. Due to a suitable regularization, the corresponding operators are correctly defined as quasi-differentials. Their resolvent convergence is investigated and all self-adjoint, maximal dissipative, and maximal accumulative extensions are described in terms of homogeneous boundary conditions of the canonical form.

1. INTRODUCTION

Many problems of mathematical physics lead to the study of Schrödinger-type operators with strongly singular (in particular distributional) potentials, see the monographs [1, 2] and the more recent papers [5, 6, 18, 19] and references therein. It should be noted that the case of very general singular Sturm-Liouville operators defined in terms of appropriate quasi-derivatives has been considered in [3] (see also the book [7] and earlier discussions of quasi-derivatives in [23, 26]). Higher-order quasi-differential operators with matrix-valued valued singular coefficients were studied in [8, 9, 21, 25].

The paper [22] started a new approach for study of one-dimensional Schrödinger operators with distributional potential coefficients in connection with such areas as extension theory, resolvent convergence, spectral theory and inverse spectral theory. The important development was achieved in [11] (see also [12, 14]), where it was considered the case of Sturm-Liouville operators generated by the differential expression

$$(1) \quad l(y) = -(py')'(t) + q(t)y(t), \quad t \in \mathcal{J}$$

with singular distributional coefficients on a finite interval $\mathcal{J} := (a, b)$. Namely it was assumed that

$$(2) \quad q = Q', \quad 1/p, \quad Q/p, \quad Q^2/p \in L_1(\mathcal{J}, \mathbb{C}),$$

where the derivative of Q is understood in the sense of distributions. The more general class of second order quasi-differential operators was recently studied in [19]. In [12, 13] two-term singular differential operators

$$(3) \quad l(y) = i^m y^{(m)}(t) + q(t)y(t), \quad t \in \mathcal{J}, \quad m \geq 2$$

with distributional coefficient q were investigated. The case of matrix operators of the form (3) was considered in [17]. Mention also [20] where the deficiency indices of matrix Sturm-Liouville operators with distributional coefficients on a half-line were studied.

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The purpose of the present paper is to extend the results of [11] to the matrix Sturm-Liouville differential expressions. In Section 2 we give a regularization of the formal differential expression (1) under a matrix analogue of assumptions (2). The question of norm resolvent convergence of such singular matrix Sturm-Liouville operators is studied in Section 3. In Section 4 we consider the case of the symmetric minimal operator and describe all its self-adjoint, maximal dissipative, and maximal accumulative extensions. In addition, we study in details the case of separated boundary conditions.

2. REGULARIZATION OF SINGULAR EXPRESSION

For positive integer s , denote by $M_s \equiv \mathbb{C}^{s \times s}$ the vector space of $s \times s$ matrices with complex coefficients. Let $\mathcal{J} := (a, b)$ be a finite interval. Consider Lebesgue measurable matrix functions p, Q on \mathcal{J} into M_s such that p is invertible almost everywhere. In what follows we shall always assume that

$$(4) \quad p^{-1}, p^{-1}Q, Qp^{-1}, Qp^{-1}Q \in L_1(\mathcal{J}, M_s).$$

This condition should be considered as a matrix (noncommutative) analogue of the assumption (2). In particular (4) is valid under the (more restrictive) condition

$$\int_{\mathcal{J}} \|p^{-1}(t)\| (1 + \|Q(t)\|^2) dt < \infty,$$

which was (locally) assumed in the above-mentioned paper [20]. Consider the block Shin–Zettl matrix

$$(5) \quad A := \begin{pmatrix} p^{-1}Q & p^{-1} \\ -Qp^{-1}Q & -Qp^{-1} \end{pmatrix} \in L_1(\mathcal{J}, M_{2s})$$

and the corresponding quasi-derivatives

$$D^{[0]}y = y, \quad D^{[1]}y = py' - Qy, \quad D^{[2]}y = (D^{[1]}y)' + Qp^{-1}D^{[1]}y + Qp^{-1}Qy.$$

For $q = Q'$ the Sturm-Liouville expression (1) is defined by

$$(6) \quad l[y] := -D^{[2]}y.$$

The quasi-differential expression (6) gives rise to the *maximal* quasi-differential operator in the Hilbert space $L_2(\mathcal{J}, \mathbb{C}^s) =: L_2$

$$L_{\max} : y \rightarrow l[y], \quad \text{Dom}(L_{\max}) := \{y \in L_2 \mid y, D^{[1]}y \in AC([a, b], \mathbb{C}^s), D^{[2]}y \in L_2\}.$$

The *minimal* quasi-differential operator is defined as a restriction of the operator L_{\max} onto the set

$$\text{Dom}(L_{\min}) := \{y \in \text{Dom}(L_{\max}) \mid D^{[k]}y(a) = D^{[k]}y(b) = 0, k = 0, 1\}.$$

Note that under the assumption

$$p^{-1}, q \in L_1(\mathcal{J}, M_s)$$

operators L_{\max}, L_{\min} introduced above coincide with the standard maximal and minimal matrix Sturm-Liouville operators. The regularization of the formally adjoint differential expression

$$l^+y := -(p^*y')'(t) + q^*(t)y(t)$$

can be defined in an analogous way (here $A^* = \overline{A^T}$ is the conjugate transposed matrix to A). Let $D^{\{k\}}$ ($k = 0, 1, 2$) be the Shin–Zettl quasi-derivatives associated with l^+ . Denote by L_{\max}^+ and L_{\min}^+ the maximal and the minimal operators generated by this expression on the space L_2 . The following results are proved in [8] (see also [21]) in the case of general quasi-differential matrix operators.

Lemma 1. (Green's formula). *For any $y \in \text{Dom}(L_{\max})$, $z \in \text{Dom}(L_{\max}^+)$ there holds*

$$\int_a^b \left(D^{[2]}y \cdot \bar{z} - y \cdot \overline{D^{[2]}z} \right) dt = (D^{[1]}y \cdot \bar{z} - y \cdot \overline{D^{[1]}z}) \Big|_{t=a}^{t=b}.$$

Lemma 2. *For any $(\alpha_0, \alpha_1), (\beta_0, \beta_1) \in \mathbb{C}^{2s}$ there exists a function $y \in \text{Dom}(L_{\max})$ such that*

$$D^{[k]}y(a) = \alpha_k, \quad D^{[k]}y(b) = \beta_k, \quad k = 0, 1.$$

Theorem 1. *The operators L_{\min} , L_{\min}^+ , L_{\max} , L_{\max}^+ are closed and densely defined in $L_2([a, b], \mathbb{C}^s)$, and satisfy*

$$L_{\min}^* = L_{\max}^+, \quad L_{\max}^* = L_{\min}^+.$$

In the case of Hermitian matrices p and Q the operator $L_{\min} = L_{\min}^+$ is symmetric with the deficiency indices $(2s, 2s)$, and

$$L_{\min}^* = L_{\max}, \quad L_{\max}^* = L_{\min}.$$

3. CONVERGENCE OF RESOLVENTS

Let $l_\varepsilon[y] = -D_\varepsilon^{[2]}y$, $\varepsilon \in [0, \varepsilon_0]$, be the quasi-differential expressions with the coefficients $p_\varepsilon, Q_\varepsilon$ satisfying (4). These expressions generate the minimal operators $L_{\min}^\varepsilon, L_{\max}^\varepsilon$ in L_2 . Consider the quasi-differential operators

$$L_\varepsilon y = l_\varepsilon[y], \quad \text{Dom}(L_\varepsilon) = \{y \in \text{Dom}(L_{\max}^\varepsilon) \mid \alpha(\varepsilon)\mathcal{Y}_\varepsilon(a) + \beta(\varepsilon)\mathcal{Y}_\varepsilon(b) = 0\}.$$

Here $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{C}^{2s \times 2s}$ be complex matrices and

$$\mathcal{Y}_\varepsilon(a) := \{y(a), D_\varepsilon^{[1]}y(a)\}, \quad \mathcal{Y}_\varepsilon(b) := \{y(b), D_\varepsilon^{[1]}y(b)\}.$$

Clearly $L_{\min}^\varepsilon \subset L_\varepsilon \subset L_{\max}^\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$. Denote by $\rho(L)$ the resolvent set of the operator L . Recall that L_ε is said to converge to L_0 in the norm resolvent sense, $L_\varepsilon \xrightarrow{R} L_0$, if there is a number $\mu \in \rho(L_0)$, such that $\mu \in \rho(L_\varepsilon)$ for all sufficiently small ε , and

$$(7) \quad \|(L_\varepsilon - \mu)^{-1} - (L_0 - \mu)^{-1}\| \rightarrow 0, \quad \varepsilon \rightarrow 0+.$$

It should be noted that if $L_\varepsilon \xrightarrow{R} L_0$, then the condition (7) is fulfilled for all $\mu \in \rho(L_0)$ (see [15]).

Theorem 2. *Suppose $\rho(L_0)$ is not empty and, for $\varepsilon \rightarrow 0+$, the following conditions hold:*

- (1) $\|p_\varepsilon^{-1} - p_0^{-1}\|_1 \rightarrow 0$,
- (2) $\|p_\varepsilon^{-1}Q_\varepsilon - p_0^{-1}Q_0\|_1 \rightarrow 0$,
- (3) $\|Q_\varepsilon p_\varepsilon^{-1} - Q_0 p_0^{-1}\|_1 \rightarrow 0$,
- (4) $\|Q_\varepsilon p_\varepsilon^{-1}Q_\varepsilon - Q_0 p_0^{-1}Q_0\|_1 \rightarrow 0$,
- (5) $\alpha(\varepsilon) \rightarrow \alpha(0), \quad \beta(\varepsilon) \rightarrow \beta(0)$,

where $\|\cdot\|_1$ is the norm in the space $L_1(\mathcal{J}, M_s)$. Then $L_\varepsilon \xrightarrow{R} L_0$.

In essential the proof of Theorem 2 repeats the arguments of [11] where the scalar case $s = 1$ was considered. Nevertheless the result seems to be new even in the case of one-dimensional Schrödinger operators with distributional matrix-valued potentials (p_ε is the identity matrix in \mathbb{C}^s). Recall the following definition [16].

Definition 1. Denote by $\mathcal{M}^m(\mathcal{J}) =: \mathcal{M}^m$, $m \in \mathbb{N}$, the class of matrix-valued functions

$$R(\cdot; \varepsilon) : [0, \varepsilon_0] \rightarrow L_1(\mathcal{J}, \mathbb{C}^{m \times m})$$

parametrized by ε such that the solution of the Cauchy problem

$$Z'(t; \varepsilon) = R(t; \varepsilon)Z(t; \varepsilon), \quad Z(a; \varepsilon) = I,$$

satisfies the limit condition

$$\lim_{\varepsilon \rightarrow 0+} \|Z(\cdot; \varepsilon) - I\|_\infty = 0,$$

where $\|\cdot\|_\infty$ is the sup-norm.

We need the following result [16].

Theorem 3. Suppose that the vector boundary-value problem

$$(8) \quad y'(t; \varepsilon) = A(t; \varepsilon)y(t; \varepsilon) + f(t; \varepsilon), \quad t \in \mathcal{J}, \quad \varepsilon \in [0, \varepsilon_0],$$

$$(9) \quad U_\varepsilon y(\cdot; \varepsilon) = 0,$$

where the matrix-valued functions $A(\cdot, \varepsilon) \in L_1(\mathcal{J}, \mathbb{C}^{m \times m})$, the vector-valued functions $f(\cdot, \varepsilon) \in L_1(\mathcal{J}, \mathbb{C}^m)$, and the linear continuous operators

$$U_\varepsilon : C(\overline{\mathcal{J}}; \mathbb{C}^m) \rightarrow \mathbb{C}^m, \quad m \in \mathbb{N},$$

satisfy the following conditions.

- 1) The homogeneous limit boundary-value problem (8), (9) with $\varepsilon = 0$ and $f(\cdot; 0) \equiv 0$ has only a trivial solution;
- 2) $A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^m$;
- 3) $\|U_\varepsilon - U_0\| \rightarrow 0, \quad \varepsilon \rightarrow 0+.$

Then, for a small enough ε , there exist Green matrices $G(t, s; \varepsilon)$ for problems (8), (9) and

$$(10) \quad \|G(\cdot, \cdot; \varepsilon) - G(\cdot, \cdot; 0)\|_\infty \rightarrow 0, \quad \varepsilon \rightarrow 0+,$$

where $\|\cdot\|_\infty$ is the norm in the space $L_\infty(\mathcal{J} \times \mathcal{J}, \mathbb{C}^{m \times m})$.

It follows from [24] that conditions (1)–(4) of Theorem 2 imply

$$A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^{2s},$$

where the block Shin–Zettl matrix $A(\cdot; \varepsilon)$ is given by the formula

$$(11) \quad A(\cdot; \varepsilon) := \begin{pmatrix} p_\varepsilon^{-1} Q_\varepsilon & p_\varepsilon^{-1} \\ -Q_\varepsilon p_\varepsilon^{-1} Q_\varepsilon & -Q_\varepsilon p_\varepsilon^{-1} \end{pmatrix}.$$

In particular $A(\cdot; 0) = A$ (see (5)). The following two lemmas reduce Theorem 2 to Theorem 3.

Lemma 3. The function $y(t)$ is a solution of the boundary-value problem

$$(12) \quad l_\varepsilon[y](t) = f(t; \varepsilon) \in L_2, \quad \varepsilon \in [0, \varepsilon_0],$$

$$(13) \quad \alpha(\varepsilon)\mathcal{Y}_\varepsilon(a) + \beta(\varepsilon)\mathcal{Y}_\varepsilon(b) = 0,$$

if and only if the vector-valued function $w(t) = (y(t), D_\varepsilon^{[1]}y(t))$ is a solution of the boundary-value problem

$$(14) \quad w'(t) = A(t; \varepsilon)w(t) + \varphi(t; \varepsilon),$$

$$(15) \quad \alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0,$$

where the matrix-valued function $A(\cdot; \varepsilon)$ is given by (11) and $\varphi(\cdot; \varepsilon) := (0, -f(\cdot; \varepsilon))$.

Proof. Consider the system of equations

$$\begin{cases} (D_\varepsilon^{[0]}y(t))' = p_\varepsilon^{-1}(t)Q_\varepsilon(t)D_\varepsilon^{[0]}y(t) + p_\varepsilon^{-1}(t)D_\varepsilon^{[1]}y(t), \\ (D_\varepsilon^{[1]}y(t))' = -Q_\varepsilon(t)p_\varepsilon^{-1}(t)Q_\varepsilon(t)D_\varepsilon^{[0]}y(t) - Q_\varepsilon(t)p_\varepsilon^{-1}(t)D_\varepsilon^{[1]}y(t) - f(t; \varepsilon). \end{cases}$$

Let $y(\cdot)$ be a solution of (12), then the definition of a quasi-derivative implies that $y(\cdot)$ is a solution of this system. On the other hand, denoting $w(t) = (D_\varepsilon^{[0]}y(t), D_\varepsilon^{[1]}y(t))$ and $\varphi(t; \varepsilon) = (0, -f(t; \varepsilon))$, we rewrite this system in the form of equation (14). Taking into account that $\mathcal{Y}_\varepsilon(a) = w(a)$, $\mathcal{Y}_\varepsilon(b) = w(b)$, one can see that the boundary conditions (13) are equivalent to the boundary conditions (15). \square

Lemma 4. *Let a Green matrix*

$$G(t, s, \varepsilon) = (g_{ij}(t, s, \varepsilon))_{i,j=1}^2 \in L_\infty(\mathcal{J} \times \mathcal{J}, \mathbb{C}^{2s \times 2s})$$

exist for the problem (14), (15) for small enough ε . Then there exists a Green function $\Gamma(t, s; \varepsilon)$ for the semi-homogeneous boundary-value problem (12), (13) and

$$\Gamma(t, s; \varepsilon) = -g_{12}(t, s; \varepsilon) \quad a.e.$$

Proof. According to the definition of a Green matrix, a unique solution of the problem (14), (15) can be written in the form

$$w_\varepsilon(t) = \int_a^b G(t, s; \varepsilon) \varphi(s; \varepsilon) ds, \quad t \in \mathcal{J}.$$

Due to Lemma 3, the latter equality can be rewritten in the form

$$\begin{cases} D_\varepsilon^{[0]}y_\varepsilon(t) = \int_a^b g_{12}(t, s; \varepsilon)(-f(s; \varepsilon)) ds, \\ D_\varepsilon^{[1]}y_\varepsilon(t) = \int_a^b g_{22}(t, s; \varepsilon)(-f(s; \varepsilon)) ds, \end{cases}$$

where $y_\varepsilon(\cdot)$ is a unique solution of (12), (13). This implies the statement of Lemma 4. \square

Proof of Theorem 2. Consider matrices

$$Q_{\varepsilon(t), \mu} = Q_\varepsilon(t) + \mu t I, \quad p_{\varepsilon(t), \mu} = p_\varepsilon(t)$$

corresponding to the operators $L_\varepsilon + \mu I$. Clearly assumption (4) and conditions (1)–(4) of Theorem 2 do not depend on μ and we can assume without loss of generality that $0 \in \rho(L_0)$. It follows that the homogeneous boundary-value problem

$$l_0[y](t) = 0, \quad \alpha(0)\mathcal{Y}_0(a) + \beta(0)\mathcal{Y}_0(b) = 0$$

has only a trivial solution. Due to Lemma 3 the homogeneous boundary-value problem

$$w'(t) = A(t; 0)w(t), \quad \alpha(0)w(a) + \beta(0)w(b) = 0$$

also has only a trivial solution. By conditions (1)–(4) of Theorem 2 we have that $A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^{2s}$, where $A(\cdot; \varepsilon)$ is given by formula (11). Thus statement of Theorem 2 implies that the problem (14), (15) satisfies conditions of Theorem 3. It follows that Green matrices $G(t, s; \varepsilon)$ of the problems (14), (15) exist. Taking into account Lemma 4 and (10) we have that

$$\begin{aligned} \|L_\varepsilon^{-1} - L_0^{-1}\| &\leq \|L_\varepsilon^{-1} - L_0^{-1}\|_{HS} = \|\Gamma(\cdot, \cdot; \varepsilon) - \Gamma(\cdot, \cdot; 0)\|_2 \\ &\leq (b-a)\|\Gamma(\cdot, \cdot; \varepsilon) - \Gamma(\cdot, \cdot; 0)\|_\infty \rightarrow 0, \quad \varepsilon \rightarrow 0+. \end{aligned}$$

Here $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. \square

Remark 1. It follows from the proof that $(L_\varepsilon - \mu)^{-1} \rightarrow (L_0 - \mu)^{-1}$ in a Hilbert-Schmidt norm for all $\mu \in \rho(L_0)$.

4. EXTENSIONS OF SYMMETRIC MINIMAL OPERATOR

In what follows we additionally suppose that the matrix functions p , Q and, consequently, the distribution $q = Q'$ to be Hermitian. By Theorem 1 the minimal operator L_{\min} is symmetric and one may consider a problem of describing (in terms of homogeneous boundary conditions) all self-adjoint, maximal dissipative, and maximal accumulative extensions of the operator L_{\min} . Let us recall following definition.

Definition 2. Let L be a closed densely defined symmetric operator in a Hilbert space \mathcal{H} with equal (finite or infinite) deficient indices. The triplet (H, Γ_1, Γ_2) , where H is an auxiliary Hilbert space and Γ_1, Γ_2 are the linear mappings of $\text{Dom}(L^*)$ onto H , is called a *boundary triplet* of the symmetric operator L , if

(1) for any $f, g \in \text{Dom}(L^*)$,

$$(L^*f, g)_{\mathcal{H}} - (f, L^*g)_{\mathcal{H}} = (\Gamma_1f, \Gamma_2g)_H - (\Gamma_2f, \Gamma_1g)_H,$$

(2) for any $f_1, f_2 \in H$ there is a vector $f \in \text{Dom}(L^*)$ such that $\Gamma_1f = f_1, \Gamma_2f = f_2$.

The definition of a boundary triplet implies that $f \in \text{Dom}(L)$ if and only if $\Gamma_1f = \Gamma_2f = 0$. A boundary triplet exists for any symmetric operator with equal non-zero deficient indices (see [10] and references therein). The following result is crucial for the rest of the paper.

Lemma 5. Triplet $(\mathbb{C}^{2s}, \Gamma_1, \Gamma_2)$, where Γ_1, Γ_2 are the linear mappings

$$\Gamma_1y := (D^{[1]}y(a), -D^{[1]}y(b)), \quad \Gamma_2y := (y(a), y(b)),$$

from $\text{Dom}(L_{\max})$ onto \mathbb{C}^{2s} is a boundary triplet for the operator L_{\min} .

Proof. According to Theorem 1, $L_{\min}^* = L_{\max}$. Due to Lemma 1,

$$(L_{\max}y, z) - (y, L_{\max}z) = \left(y \cdot \overline{D^{[1]}z} - D^{[1]}y \cdot \overline{z} \right) \Big|_a^b.$$

But

$$\begin{aligned} (\Gamma_1y, \Gamma_2z) &= D^{[1]}y(a) \cdot \overline{z(a)} - D^{[1]}y(b) \cdot \overline{z(b)}, \\ (\Gamma_2y, \Gamma_1z) &= y(a) \cdot \overline{D^{[1]}z(a)} - y(b) \cdot \overline{D^{[1]}z(b)}. \end{aligned}$$

This means that condition 1) is fulfilled. Condition 2) is true due to Lemma 2. \square

Let K be a linear operator on \mathbb{C}^{2s} . Denote by L_K the restriction of L_{\max} onto the set of functions $y \in \text{Dom}(L_{\max})$ satisfying the homogeneous boundary condition in the canonical form

$$(16) \quad (K - I)\Gamma_1y + i(K + I)\Gamma_2y = 0.$$

Similarly, L^K denotes the restriction of L_{\max} onto the set of the functions $y \in \text{Dom}(L_{\max})$ satisfying the boundary condition

$$(17) \quad (K - I)\Gamma_1y - i(K + I)\Gamma_2y = 0.$$

Clearly, L_K and L^K are the extensions of L for any K . Recall that a densely defined linear operator T on a complex Hilbert space \mathcal{H} is called *dissipative* (resp. *accumulative*) if

$$\Im(Tx, x)_{\mathcal{H}} \geq 0 \quad (\text{resp. } \leq 0), \quad \text{for all } x \in \text{Dom}(T)$$

and it is called *maximal dissipative* (resp. *maximal accumulative*) if, in addition, T has no non-trivial dissipative (resp. accumulative) extensions in \mathcal{H} . Every symmetric operator is both dissipative and accumulative, and every self-adjoint operator is a maximal dissipative and maximal accumulative one. Lemma 5 together with results of [10, Ch. 3] leads to the following description of dissipative, accumulative and self-adjoint extensions of L_{\min} .

Theorem 4. *Every L_K with K being a contracting operator in \mathbb{C}^{2s} , is a maximal dissipative extension of L_{\min} . Similarly every L^K with K being a contracting operator in \mathbb{C}^{2s} , is a maximal accumulative extension of the operator L_{\min} . Conversely, for any maximal dissipative (respectively, maximal accumulative) extension \tilde{L} of the operator L_{\min} there exists a contracting operator K such that $\tilde{L} = L_K$ (respectively, $\tilde{L} = L^K$). The extensions L_K and L^K are self-adjoint if and only if K is a unitary operator on \mathbb{C}^{2s} . These correspondences between operators $\{K\}$ and the extensions $\{\tilde{L}\}$ are all bijective.*

Remark 2. It follows from Theorem 2 and Theorem 4 that the mapping $K \rightarrow L_K$ is not only bijective but also continuous. More accurately, if contracting operators K_n converge to an operator K , then $L_{K_n} \xrightarrow{R} L_K$. The converse is also true, because the set of contracting operators in the space \mathbb{C}^{2s} is a compact set. This means that the mapping

$$K \rightarrow (L_K - \lambda)^{-1}, \quad \text{Im } \lambda < 0,$$

is a homeomorphism for any fixed λ . Analogous result is true for L^K .

Now we pass to the description of separated boundary conditions. Denote by f_a the germ of a continuous function f at the point a .

Definition 3. The boundary conditions that define the operator $L \subset L_{\max}$ are called *separated* if for arbitrary functions $y \in \text{Dom}(L)$ and any $g, h \in \text{Dom}(L_{\max})$, such that

$$g_a = y_a, \quad g_b = 0, \quad h_a = 0, \quad h_b = y_b$$

we have $g, h \in \text{Dom}(L)$.

Theorem 5. *Let K be a linear operator on \mathbb{C}^{2s} . Boundary conditions (16), (17) defining L_K and L^K respectively are separated if and only if K is block diagonal, i.e.,*

$$(18) \quad K = \begin{pmatrix} K_a & 0 \\ 0 & K_b \end{pmatrix},$$

where K_a, K_b are arbitrary $s \times s$ matrices.

Proof. We consider the operators L_K , the case of L^K can be treated in a similar way. The assumption $y_c = g_c$ implies that

$$(19) \quad y(c) = g(c), \quad (D^{[1]}y)(c) = (D^{[1]}g)(c), \quad c \in [a, b].$$

Let K have the form (18). Then (16) can be written in the form of a system,

$$\begin{cases} (K_a - I)D^{[1]}y(a) + i(K_a + I)y(a) = 0, \\ -(K_b - I)D^{[1]}y(b) + i(K_b + I)y(b) = 0. \end{cases}$$

Clearly these conditions are separated. Conversely, suppose that boundary conditions (16) are separated. The matrix $K \in \mathbb{C}^{2s \times 2s}$ can be written in the form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

We need to prove that $K_{12} = K_{21} = 0$. Let us rewrite (16) in the form of the system

$$\begin{cases} (K_{11} - I)D^{[1]}y(a) - K_{12}D^{[1]}y(b) + i(K_{11} + I)y(a) + iK_{12}y(b) = 0, \\ K_{21}D^{[1]}y(a) - (K_{22} - I)D^{[1]}y(b) + iK_{21}y(a) + i(K_{22} + I)y(b) = 0. \end{cases}$$

The fact that the boundary conditions are separated implies that a function g such that $g_a = y_a, g_b = 0$ also satisfies this system. It follows from (19) that for any $y \in \text{Dom}(L_K)$

$$\begin{cases} K_{11} [D^{[1]}y(a) + iy(a)] = D^{[1]}y(a) - iy(a), \\ K_{21} [D^{[1]}y(a) + iy(a)] = 0. \end{cases}$$

This means that for any $y \in \text{Dom}(L_K)$

$$(20) \quad D^{[1]}y(a) + iy(a) \in \text{Ker}(K_{21}).$$

For any $z = (z_1, z_2) \in \mathbb{C}^{2s}$, consider the vectors $-i(K + I)z$ and $(K - I)z$. Due to Lemma 5 and the definition of the boundary triplet, there exists a function $y_z \in \text{Dom}(L_{\max})$ such that

$$(21) \quad \begin{cases} -i(K + I)z = \Gamma_1 y_z, \\ (K - I)z = \Gamma_2 y_z. \end{cases}$$

Clearly y_z satisfies (16) and $y_z \in \text{Dom}(L_K)$. Rewrite (21) in the form of the system

$$\begin{cases} -i(K_{11} + I)z_1 - iK_{12}z_2 = D^{[1]}y_z(a), \\ -iK_{21}z_1 - i(K_{22} + I)z_2 = -D^{[1]}y_z(b), \\ (K_{11} - I)z_1 + K_{12}z_2 = y_z(a), \\ K_{21}z_1 + (K_{22} - I)z_2 = y_z(b). \end{cases}$$

The first and the third equations of the system above imply that for any $z_1 \in \mathbb{C}^s$

$$D^{[1]}y_z(a) + iy_z(a) = -2iz_1.$$

Due to (20) we have that $\text{Ker}(K_{21}) = \mathbb{C}^s$ and therefore $K_{21} = 0$. Similarly one can prove that $K_{12} = 0$. \square

Remark 3. It follows from Lemma 5 and Theorem 1 of [4] that there is a one-to-one correspondence between the generalized resolvents R_λ of L_{\min} and the boundary-value problems

$$l[y] = \lambda y + h, (K(\lambda) - I)\Gamma_1 y + i(K(\lambda) + I)\Gamma_2 y = 0.$$

Here $\text{Im } \lambda < 0$, $h \in L_2$, and $K(\lambda)$ is an operator-valued function on the space \mathbb{C}^{2s} , regular in the lower half-plane, such that $\|K(\lambda)\| \leq 1$. This correspondence is given by the identity

$$R_\lambda h = y, \quad \text{Im } \lambda < 0.$$

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